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On the dispensable role of time in games of perfect information

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Abstract In Aumann (Games Econ Behav 8(1):6–19, 1995, Games Econ Behav 23(1):97–105, 1998), time is assumed implicitly in the description of games of perfect information, and it is part of the epistemic distinction between ex-ante and ex-post knowledge. We show that ex-post knowledge in these papers can be expressed by ex-ante knowledge and therefore epistemically, time is irrelevant to the analysis. Furthermore, we show that material rationality by weak dominance and by expectation can be expressed in terms of the timeless strategic form of the game.

Keywords Common knowledge · Rationality · Perfect information

> Time present and time past
> Are both perhaps present in time future,
> And time future contained in time past.
>
> T. S. Eliot, Burnt Norton—the Four Quartets

1 Introduction

Aumann's (1987) seminal work "Correlated equilibrium as an expression of Bayesian rationality" provided analysis of games in *strategic form* in a given formally described context. By context we mean the knowledge and probabilistic beliefs of the players.

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Aumann's (1995) paper "Backward induction and common knowledge of rationality" and Aumann's (1998) "On the centipede game" did the same for games in *extensive form* of perfect information, except that unlike Aumann (1987) only knowledge of the players was involved and not their probabilistic beliefs. Our main purpose is to show that time plays less of a role in analyzing games of perfect information than Aumann (1995, 1998) imply.

The two roles of time Time, in the ordinal sense, is implicitly assumed when a game is described in extensive form: moves in the game are made, and vertices are reached one *after* the other. Strategies are defined in terms of the extensive game and hence take into account the dynamic aspect of the game. However, since the extensive form cannot be extracted from the strategic form, the latter does not fully represent the time structure of the game.[1] Time is omnipresent in the definition of the notions of *substantive rationality* studied in Aumann (1995) and *material rationality* in Aumann (1998), as these notions are defined per vertices. Moreover, these two notions can not be defined in terms of the strategic form of the game. However, we will show that a simple variations of these notions can be defined in terms of the strategic form only.

Time plays a role not only in the objective description of the game, but also in the context. The knowledge of the players changes over time since more information is acquired by the players as the game unfolds. The knowledge of the players before the game starts is called *ex-ante* knowledge. The knowledge acquired at later times is *ex-post* knowledge. Aumann (1995) studied substantive rationality which is defined in terms of ex-ante knowledge.

> A player is *substantively rational* when for each vertex v and strategy t_i of hers it is not the case that she knows *ex-ante* that t_i yields a higher payoff than her strategy.

The notion of substantive rationality differs from the standard notion of rationality, which is defined in terms of probabilistic beliefs, in that the latter is payoff relevant and behavioral, while the former is not. A notion of rationality is *payoff relevant* and *behavioral* if a player is not rational only when she could improved her real payoffs by changing her behavior. Rationality defined by maximizing expected utility, as in Aumann (1987), is payoff relevant and behavioral. Substantive rationality is neither. It requires that a player should avoid certain strategies at vertices that are not reached and cannot be reached even if the player changes her strategies. Thus, a player may fail the rationality test even though she cannot improve her real payoff by changing her behavior.

This lead to the study of material rationality in Aumann (1998), which imposes restriction on strategies only in reached vertices, and therefore is payoff relevant and

[1] Opinions are split on the importance of the extensive form for the analysis of a game. The strategic form and the extensive form of a game were considered by von Neumann and Morgenstern (1944, section 12.1.1) as "strictly equivalent", and the choice of the form that should serve as the basis of the analysis of the game, a matter of convenience. The analysis of Kohlberg and Mertens (1986) also emphasized the dispensability of the extensive form of the game. Dalkey (1953), Thompson (1952) and Elmes and Reny (1994) studied transformations of the extensive form that preserve its strategic nature and result in the strategic form. Here we study this question in the epistemic setup.

behavioral.[2] However, in defining material rationality, Aumann found it necessary to formally introduce ex-post knowledge and ex-post rationality, stating emphatically that rationality "is *inherently* ex post". Hence the definition of ex-post material rationality in Aumann (1998).

> A player is *ex-post materially rational* when for each vertex v and strategy t_i of hers, if v is reached then it is not the case that she knows *ex-post* that t_i yields a higher payoff than her strategy.

The redundancy of ex-post knowledge A strategy of a player defines which action is taken at each of her vertices. Thus, the very notion of a strategy seems to require only ex-ante knowledge. There is nothing that a player knows ex-post, when a vertex is reached, that cannot be contemplated in advance, that is, ex-ante. We first show that, indeed, adding ex-post knowledge operators for each player and each of her vertices does not enrich our language and its expressibility. Any statement that makes use of these operators can be translated into an equivalent simple statement that makes use of only the ex-ante knowledge operators of the players. In light of this, any definition of rationality can be formulated with ex-ante knowledge operators. There are no notions of rationality that can be formulated in terms of ex-post knowledge only.

In particular, applying this translation to the definition of ex-post material rationality results in the following equivalent description of ex-post material rationality. The changes from the definition are italicized.

> **Proposition:** A player is ex-post materially rational if and only if for each vertex v and strategy t_i of hers, if v is reached then it is not the case that she knows *ex-ante* that *if v is reached* then t_i yields a higher payoff than her strategy.

Thus, epistemically, time plays no role in studying games with perfect information in the model of Aumann (1995, 1998). In particular the ex-post qualification of material rationality is not justified.

The redundancy of the extensive form Time still plays a role in the objective description of the game and in the way it is used in the definition of substantive and material rationality which require rationality in vertices. Time is indeed not redundant in the definition of these two notions of rationality. However, for other notions of material rationality, which are defined too in terms of rationality in reached vertices, time is redundant.

Rationality by weak dominance Material rationality requires that there is no strategy of the player which she knows to yield a *strictly* higher payoff. Consider the following strengthening of this notion of material rationality, which requires that the player does not even know of another strategy of hers that yields payoffs which are at least as good as her strategy and is not equivalent to it.

[2] Aumann (1998) showed that common knowledge of material rationality in the centipede game implies that the first player stops the game immediately, but did not characterize the profiles that are played under this assumption in general. This has been shown by Hillas and Samet (2014) to be the set of non-probabilistic correlated equilibria.

> A player is *materially rational by weak dominance* when for each vertex v and strategy t_i of hers, it is not the case that she knows that if v is reached then t_i yields a payoff at least as high as her strategy, unless she knows that if v is reached t_i yields the same payoff as her strategy.[3]

We now write the same definition, except that we omit the word 'material' and any mention of vertices.

> A player is *rational by weak dominance* when for each strategy t_i of hers, it is not the case that she knows that t_i yields a payoff at least as high her strategy, unless she knows that that t_i yields the same payoff as her strategy.

This definition makes use only of the strategic form of the game. However, we show:

> **Proposition:** Material rationality by weak dominance is the same as rationality by weak dominance.

Thus, time, through the extensive form of the game, plays no role in material rationality by weak dominance. The difference between material rationality by weak dominance and by strong dominance stems from a simple observation. If a strategy is weakly dominated, given a player's ex-post knowledge in a *reached* vertex, then it is also weakly dominated given her ex-ante knowledge. Thus, if the player does not know ex-post at a reached vertex that her strategy is weakly dominated, then she does not also know ex-ante that it is weakly dominated. This argument is not valid for strong dominance: it is possible that a strategy is strongly dominated, given a player's ex-post knowledge at a vertex which is reached, and yet it will not be strongly dominated given her ex-ante knowledge, but only weakly dominated. This explains also why substantive rationality by weak dominance cannot be reduced to rationality in the strategic form: It is possible that a strategy is weakly dominated given a player's ex-post knowledge at a vertex which is *not* reached, and yet it will not be weakly dominated given her ex-ante knowledge

Rationality by expectation By adding probabilistic beliefs to the model, as in Aumann (1987), we can define material rationality in terms of the expected payoff of the player at reached vertices.

> A player is *materially rational by expectation* when for each vertex v and strategy t_i of hers, if v is reached, then it is not the case that she knows that conditional on reaching v, playing t_i yields her expected payoff, which is higher than her expected payoff when she plays her strategy.

Again, we write the same definition without mentioning materiality or vertices.

> A player is *rational by expectation* when for each strategy t_i of hers, it is not the case that she knows that playing t_i yields expected payoff which is higher than her expected payoff.

[3] Weak dominance here is an epistemic notion. It refers to a strategy *known* to the player to be weakly inferior to another strategy. Such an inferior strategy is weakly dominated relative to the set of profiles of the player's opponents that the player does not exclude.

This last definition is exactly the definition of rationality for games in strategic form in Aumann (1987). And for this notion of rationality too:

Proposition: Material rationality by expectation is the same as rationality by expectation.

Thus, when we change the condition of material rationality from strong dominance to expectation, we go all the way back to the timeless notion of rationality in Aumann (1987).

2 Preliminaries

We use mostly the same notations as Aumann (1995, 1998). The set of player i's vertices is denoted by V_i, and the set of i's strategies is S_i. Player i's payoff function is h_i. Knowledge is expressed in a standard partition model. The set of states is Ω. The knowledge of player i is described by a partition Π_i of Ω. The knowledge operator K_i, associated with the partition Π_i, is defined by $K_i E = \{\omega \mid \Pi_i(\omega) \subseteq E\}$, where $\Pi_i(\omega)$ is the element of Π_i that contains ω. The event CKE, that E is *common knowledge* is the event that all know E, all know that all know E and so on. The strategy profile at ω is $\boldsymbol{s}(\omega)$. For a strategy $t_i \in S_i$ and and a strategy profile s, we denote by $(s; t_i)$ the strategy profile obtained from s by replacing s_i by t_i. We assume that each player knows her strategy. This means that $\boldsymbol{s}_i$ is measurable with respect to Π_i. For a vertex v, Ω^v is the event that vertex v is reached, and h_i^v is i's payoff function in the game that starts in vertex v. We denote i's payoff function at the root by h_i.

3 Thinking ahead: ex-post turned ex-ante

3.1 Substantive and material rationality

The event that player i's strategy t_i dominates $\boldsymbol{s}_i$ at v, denoted $[t_i \succ_v \boldsymbol{s}_i]$, consists of all states ω for which $h_i^v(\boldsymbol{s}(\omega); t_i) > h_i^v(\boldsymbol{s}(\omega))$. Substantive rationality is defined in Aumann (1995) as follows.

Definition 1 The event that player i is *substantively rational* is:

$$R_i^{\mathrm{sr}} = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} \neg K_i [t_i \succ_v \boldsymbol{s}_i]. \tag{1}$$

Substantive rationality is defined in terms of one knowledge operator for each player which can be viewed as expressing the player's ex-ante knowledge, that is, knowledge of the player before the game is played. However, this notion of rationality requires that for each vertex v of the player, she does not know any strategy which is strictly better at v than the strategy she plays, even if v is not reached and the player cannot improve upon her payoffs in the game. This strong requirement was weakened in Aumann (1998) by defining ex-post materially rationality, which makes a similar requirement but only at nodes which are reached and therefore are payoff relevant.

Definition 2 The event that player i is *ex-post materially rational* is:

$$R_i^{\text{epm}} = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} \neg\Omega^v \cup \neg K_i^v[t_i \succ_v \boldsymbol{s}_i]. \tag{2}$$

Here, K_i^v is the *ex-post* knowledge operator, describing i's knowledge at the time that she learns whether vertex v is reached or not. Formally, K_i^v is the knowledge operator associated with the partition Π_i^v which is the coarsest common refinement of Π_i and the partition $\{\Omega^v, \neg\Omega^v\}$.

The reading of (2) is straightforward. Player i is ex-post materially rational when for each vertex v and strategy t_i of i, if v is reached then it is not the case that i knows ex-post at v that t_i dominates her strategy at v.[4]

The term $\neg\Omega^v$ in Definition 2 guarantees that at nodes that v is not reached there is no restriction on the strategy used by i, which distinguishes material rationality from substantive rationality. However, the use of the knowledge operators K_i^v seems to indicate that material rationality is an ex-post notion, unlike substantive rationality which is defined in terms of the ex-ante knowledge operator. Hence the qualification of this material rationality by ex-post. We show now that the event R_i^{epm} in Definition 2 can be described in terms of ex-ante knowledge, and thus material rationality, like substantive rationality, is an ex-ante notion.

3.2 Getting rid of ex-post knowledge

Ex-post knowledge is expressible in terms of ex-ante knowledge. The assertion that one knows E ex-post, after learning whether v was reached or not, is equivalent to the following assertion: Either v is reached and one knows ex-ante that if v is reached then E, or v is not reached and one knows ex-ante that if v is not reached then E. In the formal language of the model:

Proposition 1 *For each event E,*

$$K_i^v E = \big(\Omega^v \cap K_i(\neg\Omega^v \cup E)\big) \cup \big(\neg\Omega^v \cap K_i(\Omega^v \cup E)\big). \tag{3}$$

The operators K_i^v can be used as an abbreviation of the right hand side of (3), but in the case of (2) this abbreviation does not contribute to the simplicity of the definition. Using (3) for $E = [t_i \succ_v \boldsymbol{s}_i]$ and substituting in (2) results in the following simple expression for ex-post material rationality.

Corollary 1

$$R_i^{\text{epm}} = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} \neg\Omega^v \cup \neg K_i\big(\neg\Omega^v \cup [t_i \succ_v \boldsymbol{s}_i]\big). \tag{4}$$

[4] The event $\neg X \cup Y$ corresponds to the the assertion that either X does not hold, or else Y holds. But it also correspond to the assertion that if X holds then Y holds. In logic, the 'if...then...' construction, in this sense, is called material implication.

Comparing (4) with (2) we note the while K_i replaces K_i^v, the term $\neg\Omega^v$ appears now in the scope of K_i. The second event in this union says that it is not the case that player i knows *ex-ante* that *if* v *is reached* then t_i dominates $\boldsymbol{s}_i$ at v.

Thus, ex-post material rationality is simply described without ex-post knowledge operators. Epistemically, time is dispensable in both substantive rationality and material rationality.

3.3 Epistemizing material rationality

According to (4), i can be rational but fail to know it. This will be the case in a state ω, where v is not reached, i does not know that v is not reached, and she knows that when v is reached, some strategy t_i dominates her strategy at v. As $\omega \in \neg\Omega^v$, i is rational at ω. However, since she does not know $\neg\Omega^v$ there are states in $\Pi_i(\omega)$ where v is reached. As i knows in these states that t_i dominates her strategy at v, i is not rational in these states. Therefore, in some states in $\Pi_i(\omega)$, i is rational, and in some she is not. Thus, she does not know at ω that she is rational. Put differently, a player's rationality depends not only on her behavior *given her knowledge*, but also on some facts that she does not know. This diverges from standard definitions of rationality in game theory and economics. We easily fix this problem in the following definition.

Definition 3 The event that player i is *materially rational* is:

$$R_i^{\mathrm{m}} = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} (K_i \neg\Omega^v) \cup \neg K_i \left(\neg\Omega^v \cup [t_i \succ_v \boldsymbol{s}_i]\right). \tag{5}$$

The reading of (5) is simple. Player i is materially rational when for each vertex v and strategy t_i of i,

- either player i knows that v is not reached,
- or, it is not the case that player i knows that *if* v *is reached* then t_i dominates $\boldsymbol{s}_i$ at v.[5]

The reason for the first clause is this. When i knows that v is not reached, then she trivially knows that if v is reached then t_i dominates $\boldsymbol{s}_i$ at v, since the antecedent of this condition is false. Thus, without the first clause i would not be rational when she knows that a certain vertex v is not reached, which is, of course, undesirable.

The problem with (4), that a player does not know that she is rational when she is, is solved with this definition: Player i is materially rational if and only if she knows that she is materially rational, by virtue of the positive and negative introspection properties of knowledge. The relation between ex-post material rationality and material rationality is rather simple.

[5] We can read (5) alternatively as a conditional. If i does not exclude the possibility that v is reached (that is, if she does not know that v is not reached), then she does not know that if v is reached then t_i dominates $\boldsymbol{s}_i$ at v.

Proposition 2

$$R_i^{\mathrm{m}} = K_i(R_i^{\mathrm{epm}}).$$

This follows immediately from the fact that K_i is distributed over intersections, and satisfies for each E and F, $\neg K_i E = K_i \neg K_i E$, and $K_i(E \cup K_i F) = (K_i E) \cup (K_i F)$.

As we are interested in the implications of common knowledge of rationality, the difference between the two definitions is completely washed away, since in light of Proposition 2,

Corollary 2

$$CK(\cap_i R_i^{\mathrm{m}}) = CK(\cap_i R_i^{\mathrm{epm}}).$$

4 Material rationality as strategic-form rationality

The epistemic expression of time, namely, ex-post knowledge, has been shown in the previous section to play a dispensable role in studying material rationality. But time is still present in the definition of material rationality, since it is defined particularly for the extensive form of the game, using the vertices of the game tree. As we see next, the use of the extensive form of the game is peculiar to the specific definition of material rationality in terms of strong inequalities, but not to the property of materiality.

4.1 Rationality by weak dominance

Material rationality is defined in (2) and (5) in terms strong dominance, as the event $[t_i \succ_v \boldsymbol{s}_i]$ is defined by strict inequalities. We now define rationality by weak dominance, using the event $[t_i \succcurlyeq_v \boldsymbol{s}_i]$, which is the set of states ω for which $h_i^v(\boldsymbol{s}(\omega); t_i) \geq h_i^v(\boldsymbol{s}(\omega))$. Note that $[t_i \succcurlyeq_v \boldsymbol{s}_i] = [t_i \succ_v \boldsymbol{s}_i] \cup [t_i \sim_v \boldsymbol{s}_i]$, where the event $[t_i \sim_v \boldsymbol{s}_i]$ consists of the states ω for which $h_i^v(\boldsymbol{s}(\omega); t_i) = h_i^v(\boldsymbol{s}(\omega))$.

Definition 4 The event that player i is *materially rational by weak dominance* is:

$$R_i^{\mathrm{mwd}} = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} \neg K_i\left(\neg \Omega^v \cup [t_i \succcurlyeq_v \boldsymbol{s}_i]\right) \cup K_i\left(\neg \Omega^v \cup [t_i \sim_v \boldsymbol{s}_i]\right). \tag{6}$$

That is, player i is materially rational by weak dominance when for each v and t_i, if i knows that t_i weakly dominates her strategy when v is reached, then she knows that the two strategies are equivalent when v is reached.

We now define two events in terms of the strategies of the game without referring to vertices of the game tree. The event $[t_i \succcurlyeq \boldsymbol{s}_i]$ consists of all the states ω for which $h_i(\boldsymbol{s}(\omega); t_i) \geq h_i(\boldsymbol{s}(\omega))$. Similarly, $[t_i \sim \boldsymbol{s}_i]$ is the set of states ω for which $h_i(\boldsymbol{s}(\omega); t_i) = h_i(\boldsymbol{s}(\omega))$. The following definition of rationality by weak dominance is for the strategic form of the game.

Definition 5 The event that player i is *rational by weak dominance* is:

$$R_i^{\mathrm{wd}} = \bigcap_{t_i \in S_i} \neg K_i\big([t_i \not\succcurlyeq \mathbf{s}_i]\big) \cup K_i\big([t_i \sim \mathbf{s}_i]\big). \tag{7}$$

It turns out that the the use of the extensive form of the game in the definition of material rationality by weak dominance is superfluous. Material rationality of this type can be described in terms of the strategic form of the game. Time is dispensable for weak dominance rationality.

Proposition 3 *Player i is materially rational by weak dominance if and only if i is rational by weak dominance. That is,*

$$R_i^{\mathrm{mwd}} = R_i^{\mathrm{wd}}.$$

Discussion A first attempt at defining material rationality by weak dominance would replace the event $[t_i \succ_v \mathbf{s}_i]$ in (5) by the event $[t_i \not\succcurlyeq_v \mathbf{s}_i]$, and require that for each t_i and v,

$$(K_i \neg \Omega^v) \cup \neg K_i\big(\neg \Omega^v \cup [t_i \not\succcurlyeq_v \mathbf{s}_i]\big), \tag{8}$$

holds. However this definition has the following problem. By the monotonicity of knowledge, the event in (8) is a subset of $(K_i \neg \Omega^v) \cup \neg K_i\big(\neg \Omega^v \cup [t_i \sim_v \mathbf{s}_i]\big)$. But there is no reason for requiring that for player i to be rational she should not know that if v is reached then t_i is not equivalent to her strategy. Thus, we have to amend (8) by allowing a rational player to know that if v is reached t_i is equivalent to her strategy. That is, we require that for each t_i and v,

$$(K_i \neg \Omega^v) \cup \neg K_i\big(\neg \Omega^v \cup [t_i \not\succcurlyeq_v \mathbf{s}_i]\big) \cup K_i\big(\neg \Omega^v \cup [t_i \sim_v \mathbf{s}_i]\big), \tag{9}$$

Noting further that $(K_i \neg \Omega^v) \subseteq K_i\big(\neg \Omega^v \cup [t_i \sim_v \mathbf{s}_i]\big)$ we get Definition 4.

4.2 Rationality by expectation

We examine material rationality when it is expressed in terms of expectation with respect to probabilistic beliefs. Player i's beliefs are given by a type function τ_i which assigns to each state ω a probability function $\tau_i(\omega)$ on Ω called i's *type* at ω. Each type function τ_i is measurable with respect to the partition Π_i (i.e., it is constant on each element of this partition) and satisfies for each ω, $\tau_i(\Pi_i(\omega)) = 1$.[6] For simplicity, we assume *positivity*, by which we mean that for each i and ω, $\tau_i(\omega)$ is positive on $\Pi_i(\omega)$, or equivalently, that $\tau_i(\omega)(\omega) > 0$.

Player i's expected payoff given that vertex $v \in V_i$ is reached is the function $\mathbf{E}_i^v$ on Ω, defined as follows. When $\Pi_i(\omega) \cap \Omega^v \neq \emptyset$, $\mathbf{E}_i^v(\omega) = E_{\tau_i(\omega)}(h_i^v(\mathbf{s}) \mid \Omega^v)$, where

[6] In the model of knowledge and belief that we use here, the measurability of τ_i is tantamount to saying that each player knows her beliefs, and the condition $\tau_i(\Pi_i(\omega)) = 1$ is equivalent to saying that each player is certain of whatever she knows.

$E_{\tau_i(\omega)}(\cdot \mid \Omega^v)$ is the conditional expectation given Ω^v with respect to $\tau_i(\omega)$. By the positivity axiom this conditional expectation is well defined. For other ω's, $\mathbf{E}_i^v(\omega)$ is arbitrarily defined. Similarly, define $\mathbf{E}_i^v(t_i)$ by $\mathbf{E}_i^v(t_i)(\omega) = E_{\tau_i(\omega)}(h_i^v(\boldsymbol{s}; t_i) \mid \Omega^v)$, when $\Pi_i(\omega) \cap \Omega^v \neq \emptyset$ and define $\mathbf{E}_i^v(t_i)(\omega)$ arbitrarily otherwise. Analogously to Definition 3 of material rationality we define:

Definition 6 The event that player i is *materially rational by expectation* is:

$$R_i^{\text{mexp}} = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} K_i(\neg\Omega^v) \cup \neg K_i\big(\neg\Omega^v \cup [\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]\big). \tag{10}$$

As in the previous section we can define rationality by expectation in the strategic form of the game. We define $\mathbf{E}_i$ and $\mathbf{E}_i(t_i)$ as the unconditional expectation of $h_i(\boldsymbol{s})$ and $h_i(\boldsymbol{s}; t_i)$ correspondingly.

Definition 7 The event that player i is *rational by expectation* is:

$$R_i^{\text{exp}} = \bigcap_{t_i \in S_i} \neg K_i\big([\mathbf{E}_i(t_i) > \mathbf{E}_i]\big). \tag{11}$$

This is the standard definition of rationality for games in strategic form when payoffs are computed by expectation, as in Aumann (1987). Again, as in the previous subsection, material rationality turns out to be the timeless rationality in the strategic form of the game.

Proposition 4 *Player i is materially rational by expectation if and only if i is rational by expectation. That is,*

$$R_i^{\text{mexp}} = R_i^{\text{exp}}.$$

Discussion Unlike the event $[t_i \succ_v \boldsymbol{s}_i]$, in the definition of material rationality (Definition 3), the event $[\mathbf{E}_i(t_i) > \mathbf{E}_i]$ is measurable with respect to Π_i, since τ_i is measurable with respect to Π_i. This enables us to rewrite the event that a player is materially rational by expectation in a simpler way:

$$R_i^{\text{mexp}} = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} K_i(\neg\Omega^v) \cup \neg[\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]. \tag{12}$$

To see this, suppose that $\omega \in K_i\big(\neg\Omega^v \cup [\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]\big)$. Then, either $\omega \in K_i\neg\Omega^v$, or else, $\Pi_i(\omega) \cap \Omega^v \neq \emptyset$, in which case, by the definition of $\mathbf{E}_i^v$ and $\mathbf{E}_i^v(t_i)$, $\Pi_i(\omega) \subset [\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]$. Thus, $K_i\big(\neg\Omega^v \cup [\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]\big) \subseteq (K_i\neg\Omega^v) \cup K_i([\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v])$. The inverse inclusion holds by the monotonicity of K_i. Finally, as $[\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]$ is measurable with respect to Π_i, $K_i([\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]) = [\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]$. Thus $K_i\big(\neg\Omega^v \cup [\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]\big) = (K_i\neg\Omega^v) \cup [\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]$. Plugging the right hand side of this equation in (10) results in (12).

It is possible to define a weak version of material rationality by expectation analogously to material rationality by weak dominance in Definition 4:

$$\bigcap_{v\in V_i}\bigcap_{t_i\in S_i} \neg K_i\big(\neg\Omega^v \cup [\mathbf{E}_i^v(t_i) \geq \mathbf{E}_i^v]\big) \cup K_i\big(\neg\Omega^v \cup [\mathbf{E}_i^v(t_i) = \mathbf{E}_i^v]\big).$$

However, this event is the same as the event that describe material rationality by expectation in (12). Indeed, as we showed before, the first element in the union is $\neg K_i\big(\neg\Omega^v\big) \cap \neg[\mathbf{E}_i^v(t_i) \geq \mathbf{E}_i^v]$ and the second is $K_i\big(\neg\Omega^v\big) \cup [\mathbf{E}_i^v(t_i) = \mathbf{E}_i^v]$. The union of these two events is $K_i(\neg\Omega^v) \cup \neg[\mathbf{E}_i^v(t_i) \geq \mathbf{E}_i^v]\big) \cup [\mathbf{E}_i^v(t_i) = \mathbf{E}_i^v]$. But, $\neg[\mathbf{E}_i^v(t_i) \geq \mathbf{E}_i^v] \cup [\mathbf{E}_i^v(t_i) = \mathbf{E}_i^v] = [\mathbf{E}_i^v(t_i) < \mathbf{E}_i^v] \cup [\mathbf{E}_i^v(t_i) = \mathbf{E}_i^v] = \neg[\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]$.

5 Hermaphroditic rationality

In light of the redundancy of of ex-ante knowledge, we should clarify how material rationality compares to what Aumann (1998) called *ex-ante material rationality*. According to his definition, the event that player i is ex-ante materially rational is:

$$R_i^{\text{eam}} = \bigcap_{v\in V_i}\bigcap_{t_i\in S_i} \neg\Omega^v \cup \neg K_i([t_i \succ_v \boldsymbol{s}_i]). \tag{13}$$

Note that in (4) there are two occurrences of $\neg\Omega^v$, reflecting conditioning on reaching v. Omitting both results in substantive rationality. In (13), one of these occurrences is omitted: the one in the scope of K_i. As a result, this rationality is hermaphroditic; at times it is material and at other times, substantive. This is demonstrated by the following examples.

Consider a partition element of i that contains two states ω_1 and ω_2. The player's strategy in these states is s_i. Suppose that $V_i = \{v\}$, and that v is reached at ω_1 but is not reached at ω_2. Assume, moreover, that no strategy of i yields a higher conditional payoff h_i^v at ω_2, but there is a strategy t_i that yields a higher payoff h_i^v at ω_1.

Player i is substantively rational in ω_1 and ω_2, as there is no strategy that dominates s_i at v in both states. However, she is *not* materially rational, because t_i dominates s_i at the only state in which v is reached, namely, at ω_1.

As $\neg K_i([t_i \succ_v \boldsymbol{s}_i])$ holds true in the said element of the partition, i is rational in this element according to (13). Thus, here, rationality according to (13) coincides with substantive rationality. Player i wins the title of rationality by virtue of conditional payoffs at the state ω_2 *where* v *is not reached*.

Next, consider an element of i's partition where v is not reached and in which player i knows that a strategy t_i dominates her strategy at v in all the states of the element. Then, player i is not substantively rational in this element, but is materially rational, since $K_i(\neg\Omega^v)$ holds true. As $\neg\Omega^v$ holds true in both states, player i is rational according to (13). Here, rationality by (13) coincides with material rationality.

In summary, ex-post material rationality in Aumann (1998) can be defined in terms of ex-ante knowledge alone, while ex-ante material rationality there, fails the test of being material rationality.

6 Proofs

Proof of Proposition 1 By the definition of K_i^v, $\omega \in K_i^v(E)$ if and only if $\Pi_i^v(\omega) \subseteq E$. By the definition of Π_i^v, $\Pi_i^v(\omega)$ is either $\Omega^v \cap \Pi_i(\omega)$, when $\omega \in \Omega^v$, or $\neg\Omega^v \cap \Pi_i(\omega)$, when $\omega \in \neg\Omega^v$. Thus, $\omega \in K_i^v E$ if and only if either $\omega \in \Omega^v \cap \Pi_i(\omega) \subseteq E$, or $\omega \in \neg\Omega^v \cap \Pi_i(\omega) \subseteq E$. Since $\omega \in \Pi_i(\omega)$, $\omega \in \Omega^v \cap \Pi_i(\omega)$ if and only if $\omega \in \Omega^v$. Also, $\Omega^v \cap \Pi_i(\omega) \subseteq E$ if and only if $\Pi_i(\omega) \subseteq \neg\Omega^v \cup E$. Similarly, $\omega \in \neg\Omega^v \cap \Pi_i(\omega)$ if and only if $\omega \in \neg\Omega^v$, and $\neg\Omega^v \cap \Pi_i(\omega) \subseteq E$ if and only if $\Pi_i(\omega) \subseteq \Omega^v \cup E$. Hence, $\omega \in K_i^v(E)$ if and only if either $\omega \in \Omega^v$ and $\Pi_i(\omega) \subseteq \neg\Omega^v \cup E$, or $\omega \in \neg\Omega^v$ and $\Pi_i(\omega) \subseteq \Omega^v \cup E$. This is the condition for ω to be in the right hand side of (3). □

Proof of Proposition 3 Suppose $\omega_0 \in R_i^{\text{mwd}}$, and for some t_i, $\omega_0 \in K_i\big([t_i \succcurlyeq \boldsymbol{s}_i]\big)$. We need to show that $\omega_0 \in K_i\big([t_i \sim \boldsymbol{s}_i]\big)$. Suppose to the contrary that $\omega_0 \notin K_i\big([t_i \sim \boldsymbol{s}_i]\big)$. Then,

(a) for all $\omega' \in \Pi_i(\omega_0)$, $h_i(\boldsymbol{s}(\omega'); t_i) \geq h_i(\boldsymbol{s}(\omega'))$;
(b) there exists $\omega \in \Pi_i(\omega_0)$ such that $h_i(\boldsymbol{s}(\omega); t_i) > h_i(\boldsymbol{s}(\omega))$.

There must be some $v \in V_i$ such that $\omega \in \Omega^v$, or else, the path at ω is independent of i's strategy, contrary to (b). Let v be the first such vertex. Then, for any strategy s that reaches v: $(s; t_i)$ also reaches v; $h_i(s) = h_i^v(s)$; and $h_i(s; t_i) = h_i^v(s; t_i)$. We conclude by (a) that for any $\omega' \in \Pi(\omega_0) \cap \Omega^v$, $\omega' \in [t_i \succcurlyeq_v \boldsymbol{s}_i]\big)$. Hence, $\omega_0 \in K_i\big(\neg\Omega^v \cup [t_i \succcurlyeq_v \boldsymbol{s}_i]\big)$. As $\omega_0 \in R_i^{\text{mwd}}$, it follows that $\omega_0 \in K_i\big(\neg\Omega^v \cup [t_i \sim_v \boldsymbol{s}_i]\big)$. But this is a contradiction, since $\omega \in \Pi_i(\omega_0) \cap \Omega^v$ and therefore by (b), $h_i^v(\boldsymbol{s}(\omega); t_i) > h_i^v(\boldsymbol{s}(\omega))$

Conversely, suppose that $\omega_0 \in R_i^{\text{wd}}$, and for some t_i and $v \in V_i$, $\omega_0 \in K_i\big(\neg\Omega^v \cup [t_i \succcurlyeq_v \boldsymbol{s}_i]\big)$. We need to show that $\omega_0 \in K_i\big(\neg\Omega^v \cup [t_i \sim_v \boldsymbol{s}_i]\big)$. Suppose to the contrary that $\omega_0 \notin K_i\big(\neg\Omega^v \cup [t_i \sim_v \boldsymbol{s}_i]\big)$. Then,

(c) for all $\omega' \in \Pi_i(\omega_0) \cap \Omega^v$, $h_i^v(\boldsymbol{s}(\omega'); t_i) \geq h_i^v(\boldsymbol{s}(\omega'))$;
(d) there exists some $\omega \in \Pi_i(\omega_0) \cap \Omega^v$ such that $h_i^v(\boldsymbol{s}(\omega); t_i) > h_i^v(\boldsymbol{s}(\omega))$.

Let $\hat{t}_i$ be the strategy that agrees with t_i on v and all the vertices in V_i that follow v, and with $\boldsymbol{s}_i$ on all other vertices. Then, for all $\omega' \in \Pi_i(\omega_0) \cap \Omega^v$, $h_i^v(\boldsymbol{s}(\omega'); \hat{t}_i) = h_i(\boldsymbol{s}(\omega'); t_i)$, and $h_i^v(\boldsymbol{s}(\omega')) = h_i(\boldsymbol{s}(\omega'))$, and for all $\omega' \in \Pi_i(\omega_0) \cap \neg\Omega^v$, $h_i(\boldsymbol{s}(\omega'); \hat{t}_i) = h_i(\boldsymbol{s}(\omega'))$. Thus, by (c), for all $\omega' \in \Pi_i(\omega_0)$, $h_i(\boldsymbol{s}(\omega'); \hat{t}_i) \geq h_i(\boldsymbol{s}(\omega'))$. Hence, $\omega_0 \in K_i\big([\hat{t}_i \succcurlyeq \boldsymbol{s}_i]\big)$. As $\omega_0 \in R_i^{\text{wd}}$, it follows that $\omega_0 \in K_i\big([\hat{t}_i \sim \boldsymbol{s}_i]\big)$. But this is a contradiction, since $\omega \in \Pi_i(\omega_0) \cap \Omega^v$ and therefore by (d), $h_i(\boldsymbol{s}(\omega); \hat{t}_i) > h_i(\boldsymbol{s}(\omega))$. □

Proof of Proposition 4 Suppose that $\omega \notin R_i^{\text{exp}}$. Then, for some $t_i \in S_i$, $\omega \in K_i\big([\mathbf{E}_i(t_i) > \mathbf{E}_i]\big)$. There must be some vertex $v \in V_i$ such that $\Pi_i(\omega) \cap \Omega^v \neq \emptyset$, because otherwise the strategy of i cannot change her payoff, and thus $\mathbf{E}_i(t_i)(\omega') = \mathbf{E}_i(\omega')$ for all $\omega' \in \Pi_i(\omega)$ which means that $\omega \in K_i\big([\mathbf{E}_i(t_i) = \mathbf{E}_i]\big)$, contrary to our assumption. Let $\bar{V}_i$ be the set of all vertices $v \in V_i$ such that $\Pi_i(\omega) \cap \Omega^v \neq \emptyset$ and there is no $v' \in V_i$ that precede v. The events $(\Omega^v)_{v \in \bar{V}_i}$ are disjoint in pairs. For each $\omega' \in \Pi_i(\omega) \setminus \cup_{v \in \bar{V}_i} \Omega^v$, $h_i(\boldsymbol{s}; t_i) = h_i(\boldsymbol{s})$. Thus, $\mathbf{E}_i(t_i)(\omega) - \mathbf{E}_i(\omega) = \sum_{v \in \bar{V}_i} \tau_i(\Omega^v)(\mathbf{E}_i^v(t_i)(\omega) - \mathbf{E}_i^v(\omega))$. Hence, for some $v \in \bar{V}_i$, $\mathbf{E}_i^v(t_i)(\omega) - \mathbf{E}_i^v(\omega) > 0$. Thus, $\omega \in K_i(\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v)$. In addition $\omega \in \neg K_i \neg\Omega^v$, and therefore, $\omega \notin R_i^{\text{mexp}}$.

Conversely, suppose that $\omega \notin R_i^{\mathrm{mexp}}$. Then, for some $v \in V_i$ and $t_i \in S_i$, $\omega \in (\neg K_i \neg \Omega^v) \cap K_i(\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v)$. Thus, $\Pi_i(\omega) \cap \Omega^v \neq \emptyset$. Let $\hat{t}_i$ be the strategy described in the proof of Proposition 3. It follows from the properties of $\hat{t}_i$ that $\mathbf{E}_i(t_i)(\omega) - \mathbf{E}_i(\omega) = \tau_i(\Omega^v)(\mathbf{E}_i^v(\hat{t}_i)(\omega) - \mathbf{E}_i^v(\omega)) > 0$. Therefore $\omega \notin R_i^{\mathrm{exp}}$. □

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